9. Curves, Tangents, Velocity, and Acceleration

In this lecture, we will discuss

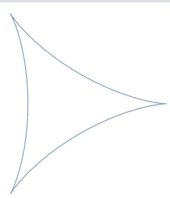
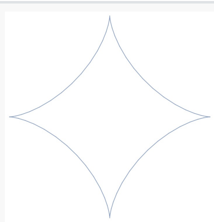
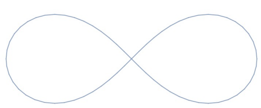
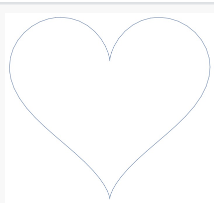
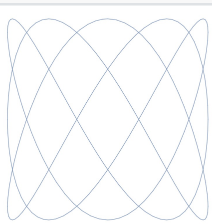
- World of Curves
  - Selected examples of curves in 2D and 3D
  - Some examples of paramatrizing a curve
- Tangents, Velocity, and Acceleration

World of Curves

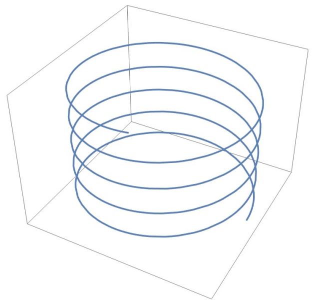
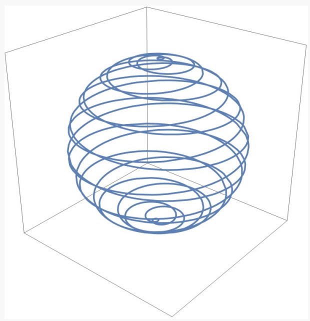
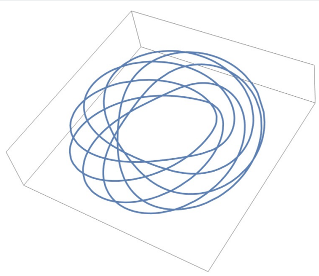
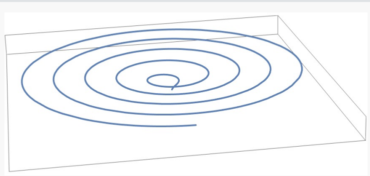
Selected examples of curves in 2D and 3D

Recall that a path is a vector-valued function of one variable, and its image, visualized as a geometric object, is a curve.

Below are some examples of curves in 2D with their equations.

Name	Curve	Parametric Equation
Deltoid		$\langle \frac{2\cos(t)}{3} + \frac{1}{3}\cos(2t) - \frac{1}{4}, \frac{2\sin(t)}{3} - \frac{1}{3}\sin(2t) \rangle, t \in [0, 2\pi]$
Astroid		$\langle \cos^3(t), \sin^3(t) \rangle, t \in [0, 2\pi]$
Lemniscate		$\langle \dfrac{\cos(t)}{\sin^2(t)+1}, \dfrac{\sin(t)\cos(t)}{\sin^2(t)+1} \rangle, t \in [0, 2\pi]$
Heart		$\langle \dfrac{4\sin^3(t)}{5}, \frac{1}{20}(13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t)) \rangle, t \in [0, 2\pi]$
Lissajous		$\langle \cos(3t)), \sin(5t) \rangle, t \in [0, 2\pi]$

Here are some examples of curves in 3D with their equations.

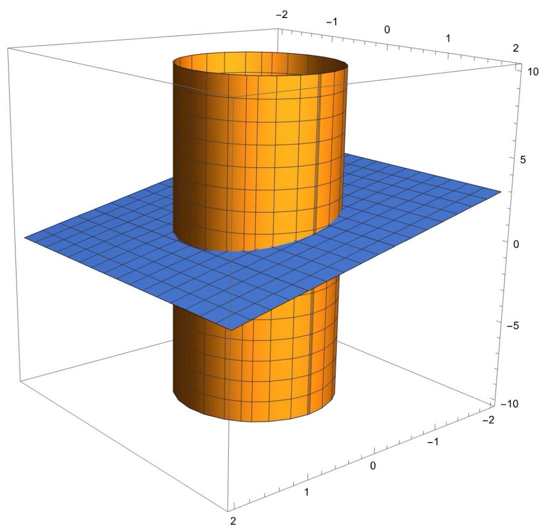
Name	Curve	Parametric Equation
Helix		$\langle 2\cos(t), 2\sin(t), \frac{t}{10}\rangle, t \in [0, 10]$
Spiral on a sphere.		$\langle \cos(t)\cos(18t), \sin(18t)\cos(t), \sin(t)\rangle, t \in [0, 2\pi]$
Torus knot		$\langle \cos(7t)(\cos(8t)+3), \sin(7t)(\cos(8t)+3), \sin(8t)\rangle, t \in [0, 2\pi]$
Spiral in 3D		$\langle t\cos(t), t\sin(t), \ln(t)\rangle, t \in [0, 10\pi]$

Some examples of paramatrizing a curve

Example 1

(1) (a) Find a vector-parametric equation $\vec{r}_1(t) = \langle x(t), y(t), z(t)\rangle$ for the shadow of the circular cylinder $x^2 + y^2 = 1$ in the xy-plane.

(2) Find a vector-parametric equation for intersection of the circular cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$ in $\mathbb{R}^3$.



ANS:

(1) Let $x(t) = \cos t$ $\quad t \in [0, 2\pi]$.

$y(t) = \sin t$

$z = z$

then we have $x(t)^2 + y^2(t) = 1$

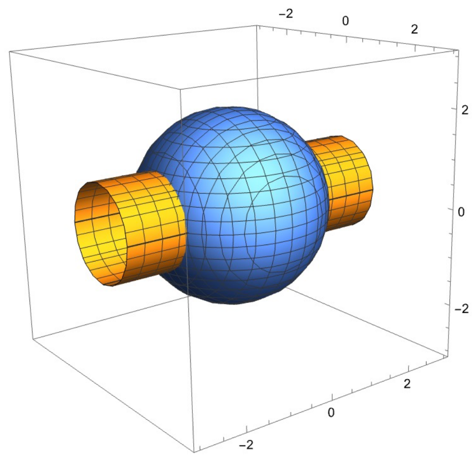
(2) We need to find functions $x(t)$, $y(t)$, $z(t)$ such that

$$\begin{cases} x^2(t) + y^2(t) = 1 \\ x(t) + z(t) = 1 \end{cases}$$

Then, one possible parametrization is

$$\vec{c}(t) = (\cos t, \sin t, 1 - \cos t), \quad t \in [0, 2\pi].$$

Example 2 Find parametric equations of the curves that are obtained by intersecting the cylinder $x^2+z^2=1$ with the sphere $x^2+y^2+z^2=4$.



ANS:

We want to find

$\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$

Since $x^2+z^2=1$,

$x(t) = \cos t$, $z(t) = \sin t$

As $x^2+y^2+z^2=4$, $\Rightarrow \cos^2 t + y^2 + \sin^2 t = 4$

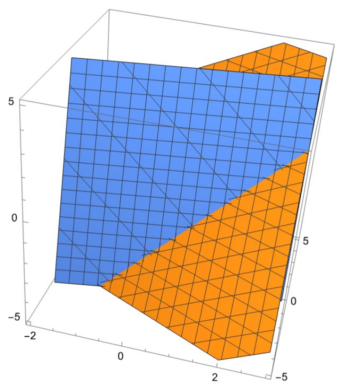
$\Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

Thus the intersection are the following two curves

$\vec{c}_1(t) = (\cos t, \sqrt{3}, \sin t)$

$\vec{c}_2(t) = (\cos t, -\sqrt{3}, \sin t)$.

Exercise 3 Find a parametric equation of the intersection of the planes $x+y-z=2$ and $2x-5y+z=3$ in $\mathbb{R}^3$.



ANS: We need to solve the system

$$\begin{cases} x+y-z=2 \\ 2x-5y+z=3. \end{cases}$$

There are two equations and three unknows, so we are allowed to choose one free value, for example, $x=t,\ t\in\mathbb{R}$. (t is a parameter)

Then we want to write y, z in terms of t.

Add $y-z=2-t$ and $-5y+z=3-2t$ together, we get

$$-4y=5-3t \Rightarrow y=-\frac{5}{4}+\frac{3}{4}t$$

Plug $y=-\frac{5}{4}+\frac{3}{4}t$ into $y-z=2-t$, we get

$$z=y-2+t=-\frac{5}{4}+\frac{3}{4}t-2+t=-\frac{13}{4}+\frac{7}{4}t.$$

Therefore, one possible parametrization is

$$\vec{c}(t)=\langle t, -\frac{5}{4}+\frac{3}{4}t, -\frac{13}{4}+\frac{7}{4}t\rangle,\ t\in\mathbb{R}$$

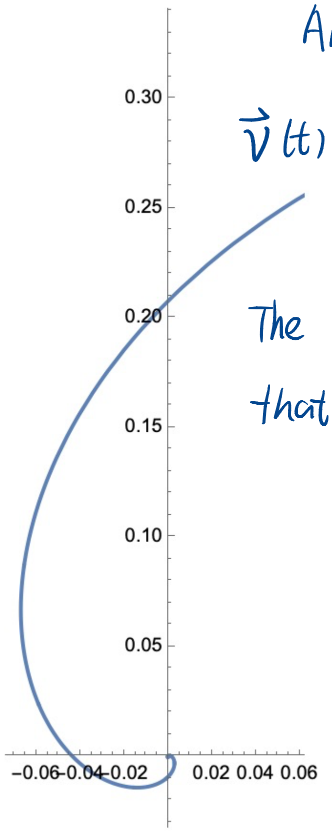
$$=\langle 0, -\frac{5}{4}, -\frac{13}{4}\rangle+t\langle 1, \frac{3}{4}, \frac{7}{4}\rangle$$

Note $\vec{c}(t)$ is a line with $(0, -\frac{5}{4}, -\frac{13}{4})$ on it with the direction by $\langle 1, \frac{3}{4}, \frac{7}{4}\rangle$.

Tangents, Velocity, and Acceleration

- The velocity of a differentiable path $\mathbf{c}(t)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is given by $\mathbf{v}(t) = \mathbf{c}'(t)$, and its speed by the scalar function $\|\mathbf{v}(t)\|$.
- If $\mathbf{c}(t)$ is twice differentiable, we define the acceleration by $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{c}''(t)$.

Example 4. The curve $\mathbf{c}(t) = \left(e^{-t}\cos t, e^{-t}\sin t\right), 0 \le t \le 3\pi$, represents the trajectory of a particle moving in $\mathbb{R}^2$. Compute its velocity and find all points at which the velocity is horizontal or vertical.



ANS: The velocity is

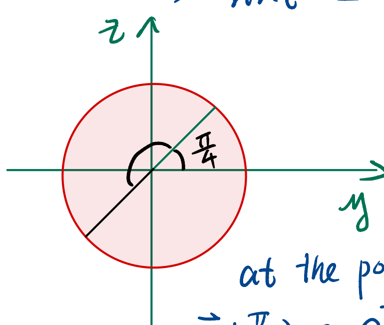
$$\vec{v}(t) = \vec{c}\,'(t) = \left(-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t\right)$$

$$= -e^{-t}\left(\cos t + \sin t,\ \sin t - \cos t\right)$$

The velocity is horizontal when its y coordinate is 0.

that is $\sin t - \cos t = 0$

$\Rightarrow \sin t = \cos t \Rightarrow \tan t = 1$.



Recall $0 \le t \le 3\pi$

Thus $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

Thus, the velocity is horizontal at the positions

$\vec{c}\left(\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\vec{c}\left(\frac{5\pi}{4}\right) = -e^{-\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\vec{c}\left(\frac{9\pi}{4}\right) = e^{-\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Similarly, we set x cooridinate to be zero to find t such that the velocity is vertical.

that is $\sin t = -\cos t$, $\Rightarrow \tan t = -1$

So we have $t = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

Thus the velocity is vertical at positions

$\vec{c}\left(\frac{3\pi}{4}\right) = e^{-\frac{3\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\vec{c}\left(\frac{7\pi}{4}\right) = -e^{-\frac{5\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\vec{c}\left(\frac{9\pi}{4}\right) = e^{-\frac{9\pi}{4}}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

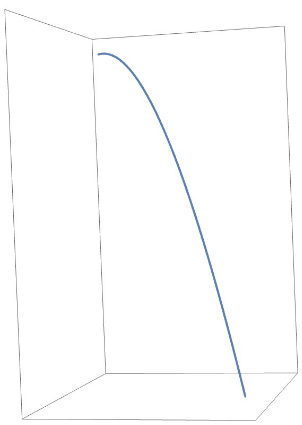
Example 5. A stone is thrown from a rooftop at time $t = 0$ seconds. Its position at time t is given by $\mathbf{r}(t) = 4t\boldsymbol{i} - 5t\boldsymbol{j} + (9.8 - 4.9t^2)\boldsymbol{k}$. The origin is at the base of the building, which is standing on flat ground. Distance is measured in meters. The vector $\boldsymbol{i}$ points east, $\boldsymbol{j}$ points north, and $\boldsymbol{k}$ points up.

(1) How high is the rooftop?

(2) When does the stone hit the ground?

(3) Where does the stone hit the ground?

(4) How fast is the stone moving when it hits the ground?



ANS: (1) At $t=0$, $\vec{r}(0) = 9.8\vec{k}$. it is 9.8 m. from the ground.

(2). The stone hits the ground when

$$9.8 - 4.9t^2 = 0 \Rightarrow t = \sqrt{2}$$

(3) At $t=\sqrt{2}$, we have

$$\vec{r}(\sqrt{2}) = 4\sqrt{2}\,\vec{i} - 5\sqrt{2}\,\vec{j} + 0\vec{k}$$
$$\approx 5.6568\,\vec{i} - 7.07107\,\vec{j}$$

(4) The question is asking us the value $\|\vec{v}(t)\|$ at $t=\sqrt{2}$.

Note $\vec{v}(t) = \vec{r}\,'(t) = 4\vec{i} - 5\vec{j} - 9.8t\vec{k}$

and $\|\vec{v}(\sqrt{2})\| = \sqrt{4^2 + 5^2 + (9.8\sqrt{2})^2}$

≈ 15.267 m/s

Example 6.

(1) Compute the starting and ending positions (at times $t=0$ and $t=1$, respectively) for the path of motion described by the vector-valued function: $\mathbf{h}(t) = \left(\sin(t), \dfrac{4t-5}{3t+4}, \cos(t)\right)$.

(2) Compute the derivative of $\mathbf{h}(t)$.

(3) Compute the starting and ending velocities for $\mathbf{h}(t)$.

ANS: (1) The starting position

$$\vec{h}(0) = (0, -1.25, 1)$$

The ending position is

$$\vec{h}(1) = (0.841471, -0.142857, 0.540302)$$

(2) $$h'(t) = \left(\cos t, \frac{4(3t+4)-3(4t-5)}{(3t+4)^2}, -\sin t\right)$$

$$h'(0) = (1, 1.9375, 0)$$

$$h'(1) = (0.540302, 0.632653, -0.841471)$$

Exercise 7. Let $\mathbf{r}(t) = \langle t^2, 1-t, 4t \rangle$. Calculate the derivative of $\mathbf{r}(t) \cdot \mathbf{a}(t)$ at $t = 6$, assuming that $\mathbf{a}(6) = \langle 3, 7, -3 \rangle$ and $\mathbf{a}'(6) = \langle -7, -5, -6 \rangle$.

Solution: By the Product Rule for dot products we have

$$\frac{d}{dt}\mathbf{r}(t) \cdot \mathbf{a}(t) = \mathbf{r}(t) \cdot \mathbf{a}'(t) + \mathbf{r}'(t) \cdot \mathbf{a}(t)$$

At $t = 6$ we have

$$\frac{d}{dt}\mathbf{r}(t) \cdot \mathbf{a}(t)\bigg|_{t=6} = \mathbf{r}(6) \cdot \mathbf{a}'(6) + \mathbf{r}'(6) \cdot \mathbf{a}(6)$$

We compute the derivative $\mathbf{r}'(6)$:

$$\mathbf{r}'(t) = \frac{d}{dt}\langle t^2, 1-t, 4t \rangle = \langle 2t, -1, 4 \rangle \Rightarrow \mathbf{r}'(6) = \langle 12, -1, 4 \rangle$$

Also, $\mathbf{r}(6) = \langle 6^2, 1-6, 4 \cdot 6 \rangle = \langle 36, -5, 24 \rangle$.

Substituting the vectors in the equation above, we obtain:

$$\begin{aligned} &\frac{d}{dt}\mathbf{r}(t) \cdot \mathbf{a}(t)\bigg|_{t=6} = \langle 36, -5, 24 \rangle \cdot \langle -7, -5, -6 \rangle + \langle 12, -1, 4 \rangle \cdot \langle 3, 7, -3 \rangle \\ &= (-252 + 25 - 144) + (36 - 7 - 12) = -354 \end{aligned}$$

The derivative of $\mathbf{r}(t) \cdot \mathbf{a}(t)$ at $t = 6$ is -354 .